Double-logarithmic finite-size scaling behavior of the specific heat does not necessarily imply its divergence

Jae-Kwon Kim
School of Physics, Korea Institute for Advanced Study, 207-43, Cheongryangri-Dong, Dongdaemun-Gu, Seoul 130-012, Korea

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In a recent paper [Phys. Rev. E **60**, 3823 (1999)] Mazzeo and Kühn presented very careful analyses of the finite-size scaling (FSS) data obtained from the numerical transfer matrix technique. In this report, as a complement to that paper, a critical argument based on numerical evidence is given against the conventional interpretation of the FSS behavior of the specific heat. The argument is likely to cast doubt on previous claims for numerical evidence for the scenario of a logarithmic correction.

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The critical behavior of the two-dimensional (2D) disordered Ising ferromagnet has been a controversial issue both theoretically and numerically. There are currently two main scenarios, namely, the scenario of weak universality versus that of the logarithmic correction. In spite of the existence of extensive numerical efforts, it was correctly pointed out by Mazzeo and Kühn [1] (MK) that most previous finite-size scaling (FSS) analyses can be interpreted ambivalently.

From various Monte Carlo studies on the disordered Ising ferromagnet [2,3], it is now well known that the specific heat at the critical point *increases* mildly as a function of the linear size of the lattice. This mild dependence can be expressed as a double-logarithmic function, that is,

$$C_L \sim C_0' + C_1' \ln(1 + C_2' \ln L), \tag{1}$$

where C_1' and C_2' should be positive for increasing C_L. Most authors have interpreted Eq. (1) as evidence for the critical behavior of the double-logarithmic divergence [2,4,5]. MK argued and showed that their transfer matrix data fitted to Eq. (1) can as well be fitted to

$$C_L \sim C_\infty + C_3' L^{\alpha/\nu}, \tag{2}$$

with negative values of α/ν and C_3' and with a positive nonzero value of C_∞. Equation (2) is an indication of the validity of the scenario of weak universality.

In this report, instead of showing the ambivalence of the fits, we take a different approach to demonstrate that Eq. (1) is unlikely to imply the scaling behavior of the logarithmic divergence. To this end we start with a model where the value of α is explicitly known to be negative. One such model mostly studied by field theory [7], series expansion methods [8], and Monte Carlo methods [9] is the simple cubic N-vector model with $N=3$ (the nearest neighbor Heisenberg model). For this model the values of α and the critical temperature are $\alpha \simeq -0.11(3)$ and $T_c \simeq 1.4430$, respectively [7–9].

At T_c, we measured the specific heat (C_L) of the Heisenberg model by a Monte Carlo method based on Wolff's single cluster algorithm. The measurements are over the range of L up to $L=120$ (1 728 000 lattice sites) with the periodic boundary condition imposed on the simple cubic lattice.

We plot our data for the specific heat against $\ln(\ln L)$ (Fig. 1). The figure clearly shows that the data scale nicely with $\ln(\ln L)$ up to $L=60$ but start to scale much *faster* than that with increasing values of L. Consequently, the data do not fit Eq. (1) but fit Eq. (2) with the value of $\alpha/\nu \simeq 0.237$. This estimate is unacceptable in light of the conventionally accepted estimate, that is, $\alpha/\nu \simeq -0.15$.

A feature emerging from this analysis is that for the specific heat the true asymptotic FSS behavior may manifest itself for values of L much larger than for other typical variables like the correlation length and the magnetic susceptibility, and thus the preasymptotic FSS behavior of the specific heat may yield a grossly misleading estimate. We also deduce from the figure another feature: the FSS behavior of the specific heat is nonuniform as a function of L since it must converge in the thermodynamic limit. We stress that these features hold generally regardless of the value of α: For the 2D three-state Potts model where the exact value of α is positive, the FSS behavior of the specific heat is nonuniform and its FSS analysis does not yield an accurate estimate of α/ν while that of other variables does [10].

This result is remarkable because for the disordered Ising

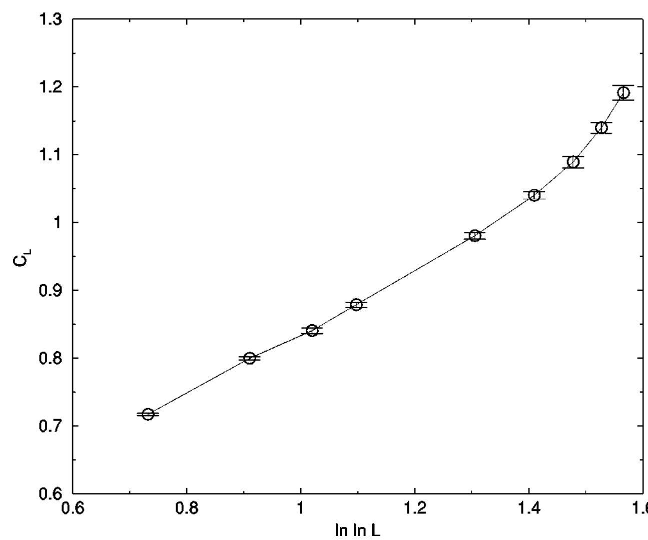

FIG. 1. $C_L(T_c)$ versus $\ln(\ln L)$ for the simple cubic Heisenberg ferromagnet. Here $T_c = 1.4430$ and the range of L used is $8 \leq L \leq 120$.

ferromagnet the specific heat data appeared to be consistent with the $\ln(\ln L)$ behavior [2–5]. Even though one assumes that the data sets are in the asymptotic regime, this consistency cannot be regarded as conclusive due to the aforementioned ambivalence of the fits. On the other hand, if the data sets are not in the asymptotic regime, a FSS analysis of the specific heat can yield a completely meaningless estimate even *in the absence* of any ambiguity in the interpretation. It is thus very unlikely that the $\ln(\ln L)$ behavior can be regarded as conclusive evidence for the scenario of the logarithmic correction. It is also clear from the nonuniformity that the preasymptotic FSS behavior of the specific heat between $\ln L$ and $\ln(\ln L)$ found in Ref. [6] should not be interpreted as crossover behavior.

At this point one may wonder if our result is an artifact of a tiny error in T_c. We actually observed similar behavior for all the temperature we tried near T_c. Moreover, there are other Monte Carlo data reported at $T_c = 1.4427$ up to $L = 32$ [11], which also show seemingly divergent FSS behavior of the specific heat.

We also remark that similar FSS behavior (faster than the double-logarithmic FSS) was obtained at the critical point of the 3D N-vector model for $N = 5$ [12], where the value of the critical exponent of the correlation length is larger than that of the Heisenberg model [7]. All those cross-checks certainly rule out the possibility of an artifact.

The ambivalence in interpretation is not limited to analysis of the FSS behavior only; it also appears in analysis of thermodynamic data. In the Monte Carlo method the thermodynamic data of such physical variables as the magnetic susceptibility and the correlation length at a given temperature can be properly obtained under the *thermodynamic condition* that the linear size of the lattice L is much larger than the corresponding correlation length ξ_L, e.g., $L/\xi_L \gtrsim 7$ for typical 2D *undisordered* systems [13]. Provided that a set of thermodynamic data measured at various different temperatures in the scaling regime is available, the data set that can be fitted to a pure power-law singularity can equally well be fitted to a modified power law with some multiplicative logarithmic correction. To overcome this ambivalence measurements of the thermodynamic data are needed over too extremely broad a range for the usual Monte Carlo measurement to even be possible. In Ref. [3], however, the *thermodynamic* data for the specific heat in the scaling regime, not the FSS data at criticality, were observed to be decreasing with decreasing temperature toward the critical point. The observed finite peak in the thermodynamic specific heat at a noncritical temperature can by no means be accounted for in the context of a scenario of a logarithmic correction.

A series expansion study does not suffer from the finite-size effect, so the aforementioned ambiguity problem in the analysis of FSS does not apply to an analysis of the series expansion. Nevertheless, a series expansion study suffers from the same ambivalence problem encountered in the analysis of the thermodynamic data: It was shown in a recent high-temperature expansion study of the 2D random bond disordered Ising ferromagnet [14] that, for a given strength of disorder, both a pure power-law singularity and a modified power-law singularity with a logarithmic correction can be consistent with the series expansion. An odd claim from the series expansion study is that, while the critical exponent γ obtained from the assumption of the pure power-law singularity increases monotonically with increasing strength of the disorder, the critical exponent of the logarithmic term from the assumption of the modified power-law singularity does not change accordingly [14]. A more recent paper [15] suggested that the claim is unlikely to be correct. It was also clearly demonstrated that, for a sufficiently strong disorder, the critical exponent of the logarithmic term is *definitely* larger than the proposed value in the scenario of a logarithmic correction [15].

To conclude, we have provided an example where the value of α is negative, yet the FSS behavior looks more divergent than double-logarithmic behavior. This serves as a clear counterexample against the conventional interpretation of the FSS behavior of the specific heat made in some studies of the critical behavior of the 2D disordered Ising ferromagnet.

[1] G. Mazzeo and R. Kühn, Phys. Rev. E **60**, 3823 (1999).
[2] V.B. Andreichenko, Vl.S. Dotsenko, W. Selke, and J.-S. Wang, Nucl. Phys. B **344**, 531 (1990).
[3] J.-K. Kim and A. Patrascioiu, Phys. Rev. Lett. **72**, 2785 (1994); Phys. Rev. B **49**, 15 764 (1994).
[4] H.G. Ballesteros, L.A. Fernández, V.M. Mayor, A.M. Sudupe, G. Parisi, and J.J. Ruiz-Lorenzo, J. Phys. A **30**, 8379 (1997).
[5] W. Selke, L.N. Shchur, and O.A. Vasiliev, Physica A **259**, 388 (1998).
[6] D. Stauffer, F.D.A. Aarao Reis, S.L.A. Queiroz, and R.R. dos Santos, Int. J. Mod. Phys. C **8**, 1209 (1997).
[7] R. Guida and J. Zinn-Justin, J. Phys. A **31**, 8103 (1998); S.A. Antonenko and A.I. Sokolov, Phys. Rev. E **51**, 1894 (1995); H. Kleinert, e-print cond-mat/9812197.
[8] P. Butera and M. Comi, Phys. Rev. B **56**, 8212 (1997).
[9] K. Chen, A.M. Ferrenberg, and D.P. Landau, Phys. Rev. B **48**, 3249 (1993); C. Holm and W. Janke, J. Phys. A **27**, 2553 (1994).
[10] J.-K. Kim and D.P. Landau, Physica A **250**, 362 (1998).
[11] R.G. Brown and M. Ciftan, Phys. Rev. Lett. **76**, 1352 (1996).
[12] J.-K. Kim (unpublished).
[13] See, for example, J.-K. Kim, A.J.F. de Souza, and D.P. Landau, Phys. Rev. E **54**, 2291 (1996), and references therein.
[14] A. Roder, J. Adler, and W. Janke, Physica A **265**, 28 (1998).
[15] J.-K. Kim, Phys. Rev. B **61**, 1246 (2000).